J Glob Optim (2007) 38:119–127
DOI 10.1007/s10898-006-9086-0

ORIGINAL ARTICLE

A (CF)-mapping of integral functional of locally lipschitz functions

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Received: 18 December 2005 / Accepted: 18 August 2006 / Published online: 4 October 2006

Abstract We find a (CF)-mapping of the integral functional of locally Lipschitz functions f_t parametrized by $t \in T$. In the process of obtaining a (CF)-mapping, the hypothesis of upper semicontinuity of the set-valued map $t \mapsto C_{f_t}(x)$ is needed, where $C_{f_t}(x)$ denotes a convexificator of f_t at x. As a corollary of our result, we get (CF)-mappings which are obtained by Clarke subdifferentials and Michel–Penot subdifferentials, respectively. Finally, the examples specifically deriving a convexificator of the integral functional are provided.

Keywords Convexificator · (CF)-mapping · integral functional

Mathematics Subject Classification (2000) 49J52

1 Introduction and preliminaries

The development of generalized subdifferential, namely convexificator in nonsmooth analysis provides sharp extremality conditions and good calculus rules for nonsmooth functions. The idea of convexificators has been used to extend, unify and sharpen various results in nonsmooth analysis and optimization.

In this paper, we find convexificators of the integral functional of locally Lipschitz functions. As a corollary of our result, we give convexificators of the integral functional, which are obtained by Clarke subdifferentials and Michel–Penot subdifferentials, respectively.

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Let the function $f\colon \Omega \to \mathbb{R}$ be locally Lipschitz on an open set $\Omega \subset \mathbb{R}^n$. Then its *upper and lower Dini directional derivatives*

$$f^{\uparrow}(x,d) = \limsup_{\lambda \downarrow 0} \frac{f(x+\lambda d) - f(x)}{\lambda}$$

and

$$f^{\downarrow}(x,d) = \liminf_{\lambda \downarrow 0} \frac{f(x+\lambda d) - f(x)}{\lambda}$$

are also Lipschitz as functions of the direction d.

The *upper and lower Clarke directional derivatives* are defined as follows (see [2,5]):

$$f_{cl}^{\uparrow}(x,d) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{f(x'+\lambda d) - f(x')}{\lambda} \tag{1}$$

and

$$f_{cl}^{\downarrow}(x,d) = \liminf_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{f(x'+\lambda d) - f(x')}{\lambda}. \tag{2}$$

Since f is locally Lipschitz, the limits in (1) and (2) exist and are finite, and the following properties hold:

$$f_{cl}^{\uparrow}(x,d) = \max_{v \in \partial_{cl} f(x)} \langle v, d \rangle$$

and

$$f_{cl}^{\downarrow}(x,d) = \min_{w \in \partial_{cl} f(x)} \langle w, d \rangle ,$$

where

$$\partial_{cl} f(x) = co \left\{ v \in \mathbb{R}^n \mid \exists \{x_i\}\colon x_i \to x,\, x_i \in T(f),\, f'(x_i) \to v \right\}$$

and $T(f)$ is the set of points of Ω where f is differentiable. The set $\partial_{cl} f(x)$, called *Clarke subdifferential* of f at x, is a nonempty, convex and compact set. We note that the relation that the Clarke subdifferential satisfies is usually known as the *strong convexificator condition*.

Michel and Penot proposed the following generalized derivatives (see [6]):

$$f_{mp}^{\uparrow}(x,d) = \sup_{q \in \mathbb{R}^n} \left\{ \limsup_{\lambda \downarrow 0} \frac{f\left(x+\lambda(d+q)\right) - f(x+\lambda q)}{\lambda} \right\} \tag{3}$$

and

$$f_{mp}^{\downarrow}(x,d) = \inf_{q \in \mathbb{R}^n} \left\{ \liminf_{\lambda \downarrow 0} \frac{f\left(x+\lambda(d+q)\right) - f(x+\lambda q)}{\lambda} \right\}. \tag{4}$$

We call (3) and (4) the *upper and lower Michel–Penot directional derivative* of f at x in the direction d, respectively. Since f is locally Lipschitz, there exists a nonempty, convex and weak*-compact set $\partial_{mp}f(x)$, called *Michel–Penot subdifferential* of f at x, and the following properties hold:

$$f_{mp}^{\uparrow}(x,d) = \max_{v\in\partial_{mp}f(x)} \langle v,d\rangle$$

and

$$f_{mp}^{\downarrow}(x,d) = \min_{w\in\partial_{mp}f(x)} \langle w,d\rangle\,.$$

Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be a positively homogeneous function of degree 1. Demyanov introduced the concept of convexificator (see [3]). A convex compact set $C \subset \mathbb{R}^n$ is a *convexificator* (CF) of h if

$$\min_{w\in C}\langle w,d\rangle \le h(d) \le \max_{v\in C}\langle v,d\rangle\,,\quad \forall d\in\mathbb{R}^n.$$

We get

$$\min_{w\in\partial_{cl}f(x)} \langle w,d\rangle \le f^{\downarrow}(x,d) \le f^{\uparrow}(x,d) \le \max_{v\in\partial_{cl}f(x)} \langle v,d\rangle\,.$$

Hence, the Clarke subdifferential of f at x is a convexificator of both functions

$$h(d)=f^{\uparrow}(x,d) \qquad \text{and} \quad h(d)=f^{\downarrow}(x,d).$$

Also we have

$$\min_{w\in\partial_{mp}f(x)} \langle w,d\rangle \le f^{\downarrow}(x,d) \le f^{\uparrow}(x,d) \le \max_{v\in\partial_{mp}f(x)} \langle v,d\rangle\,.$$

Thus, the Michel–Penot subdifferential of f at x is also a convexificator of both functions

$$h(d)=f^{\uparrow}(x,d) \qquad \text{and} \quad h(d)=f^{\downarrow}(x,d).$$

We call a convexificator $C^+(x)$ ($C^-(x)$) of the function $h(d) = f^{\uparrow}(x,d)$ ($f^{\downarrow}(x,d)$) an *upper* (*lower*) *convexificator* of f at x. If $C(x)$ is a convexificator of both functions $f^{\uparrow}(x,d)$ and $f^{\downarrow}(x,d)$, we say that $C(x)$ is a *convexificator* of f at x. If a function f is quasidifferentiable at a point x, we can construct a convexificator and study the directional derivative by means of this convexificator. Also we can get the condition for an extremum by convexificator (see [4]).

Now we define a (CF)-mapping for a locally Lipschitz function, introduced by Demyanov and Jeyakumar ([4]). A mapping C^+ (C^-)$\colon \Omega \to 2^{\mathbb{R}^n}$ is an *upper* (*lower*) (CF)-*mapping* of f on Ω if for every $x \in \Omega$ the upper (lower) convexificator $C^+(x)$ ($C^-(x)$) satisfies the following inequalities

$$\min_{w\in C^+(x)} \langle w,d\rangle \le f^{\uparrow}(x,d) \le \max_{v\in C^+(x)} \langle v,d\rangle\,,\quad \forall d\in\mathbb{R}^n$$

$$\left(\min_{w\in C^-(x)} \langle w,d\rangle \le f^{\downarrow}(x,d) \le \max_{v\in C^-(x)} \langle v,d\rangle \qquad \forall d\in\mathbb{R}^n\right).$$

A mapping $C\colon \Omega \to 2^{\mathbb{R}^n}$ is called a (CF)-*mapping* of f if for every $x \in \Omega$ the convexificator $C(x)$ satisfies the inequalities

$$\min_{w\in C(x)} \langle w,d\rangle \le f^{\downarrow}(x,d) \le f^{\uparrow}(x,d) \le \max_{v\in C(x)} \langle v,d\rangle\,,\quad \forall d\in\mathbb{R}^n.$$

Finally, we need the following preliminaries for the proof of our main result. Let $\Gamma\colon T \to 2^{\mathbb{R}^n}$ be a set-valued map, where $T \subset \mathbb{R}$. We define Γ to be *upper semicontinuous at* $t \in T$ if given $\varepsilon > 0$, there is a $\delta > 0$ such that for all t' in T with $|t - t'| < \delta$ we have

$$\Gamma(t') \subset \Gamma(t) + \varepsilon B,$$

where B is the open unit ball in $\mathbb{R}^n$. We assume that $(T, \mathcal{T}, \mu)$ is a measure space, where $T \subset \mathbb{R}$, $\mathcal{T}$ is the σ-algebra of Borel sets $\subset T$, μ is a measure on $(T, \mathcal{T})$ (which is not necessarily the Lebesgue measure), and that f_i and g are measurable real-valued functions on T. Then we have

Theorem 1 (See 12, p93 from [7]). *Let g be an integrable function on T, and assume that $\{f_i\}$ is a sequence of measurable functions such that $|f_i(x)| \le g(x)$ on T. Then we have*

$$\int_T \liminf_i f_i \le \liminf_i \int_T f_i \le \limsup_i \int_T f_i \le \int_T \limsup_i f_i.$$

2 Main result

Let $f_t\colon \Omega \to \mathbb{R}$ be a family of Lipschitz functions of rank $k(t)$, where $t \in T \subset \mathbb{R}$ and Ω is an open set $\subset \mathbb{R}^n$. We assume that the map $t \mapsto f_t(x)$ is measurable for each x in Ω, and that k is in the space $L^1(T, \mathbb{R})$ of integrable functions from T to $\mathbb{R}$.

Lemma 1 *The maps $t \mapsto f_t^{\uparrow}(x, d)$ and $t \mapsto f_t^{\downarrow}(x, d)$ are measurable for each $(x, d) \in \Omega \times \mathbb{R}^n$.*

Proof Since f_t is continuous on Ω, we can choose a sequence $\lambda_i \downarrow 0$ such that $f_t^{\uparrow}(x, d)$ is the lim sup of

$$\frac{f_t(x + \lambda_i d) - f_t(x)}{\lambda_i}. \tag{5}$$

But (5) defines a measurable function of t because the map $t \mapsto f_t(x)$ is measurable for each x in Ω. Hence $f_t^{\uparrow}(x, d)$, as the countable lim sup of measurable functions of t, is measurable in t. Similarly, $f_t^{\downarrow}(x, d)$, as the countable lim inf of measurable functions of t, is measurable in t. □

We define the *integral functional* $f : \Omega \to \mathbb{R}$ by

$$f(x) = \int_T f_t(x)\mu(\,\mathrm{d}t)$$

and assume that f is well-defined at some point x_0 in Ω. Then we have the following lemma.

Lemma 2 *f is well-defined and Lipschitz of rank K on Ω, where $K = \int_T k(t)\mu(\mathrm{d}t)$.*

Proof Let x be any point in Ω. Then, since the map $t \mapsto f_t(x)$ is measurable, we have

$$\begin{aligned}|f(x) - f(x_0)| &\leq \int_T |f_t(x) - f_t(x_0)|\mu(\mathrm{d}t) \\ &\leq \int_T k(t)\|x - x_0\|\mu(\mathrm{d}t) \\ &= K\|x - x_0\|.\end{aligned}$$

Hence, f is well-defined and Lipschitz of rank K on Ω. □

Theorem 2 *Let $C_{f_t}(x) \subset \prod_n[-k(t), k(t)]$ be a convexificator of f_t at x, where k is in $L^1(T, \mathbb{R})$. Assume that the set-valued map $t \mapsto C_{f_t}(x)$ is upper semicontinuous on T. Then*

$$C_f(x) = \overline{co} \int_T C_{f_t}(x)\mu(\mathrm{d}t)$$

is a convexificator of f at x and the set-valued map $x \mapsto C_f(x)$ is a (CF)*-mapping of f, where $\overline{co}$ denotes the closed convex hull.*

Remark 1 We interpret $\int_T C_{f_t}(x)\mu(dt)$ as follows:

To every v in $\int_T C_{f_t}(x)\mu(dt)$, there corresponds a mapping $t \mapsto v_t$ from T to $C_{f_t}(x)$ such that the function $t \mapsto \langle v_t, d\rangle$ belongs to $L^1(T, \mathbb{R})$ and

$$\langle v, d\rangle = \int_T \langle v_t, d\rangle \, \mu(\mathrm{d}t) \tag{6}$$

for each d in $\mathbb{R}^n$. That is, every v in $\int_T C_{f_t}(x)\mu(dt)$ is an element of $\mathbb{R}^n$ that can be written as (6), where the map $t \mapsto v_t$ is a measurable selection of $C_{f_t}(x)$.

Remark 2 We note that $\int_T C_{f_t}(x)\mu(dt)$ is bounded in $\mathbb{R}^n$, because for every v in $\int_T C_{f_t}(x)\mu(dt)$, the following inequalities hold:

$$\begin{aligned}|\langle v, d\rangle| &\leq \int_T |\langle v_t, d\rangle| \, \mu(\mathrm{d}t) \\ &\leq \sqrt{n}\int_T k(t)\mu(\mathrm{d}t) \cdot \|d\| \\ &= \sqrt{n}K\|d\|.\end{aligned}$$

Remark 3 The assumption of the upper semicontinuity of the set-valued map $t \mapsto C_{f_t}(x)$ guarantees the measurability of the functions

$$t \mapsto \max_{v_t \in C_{f_t}(x)} \langle v_t, d\rangle\,, \quad t \mapsto \min_{w_t \in C_{f_t}(x)} \langle w_t, d\rangle\,, \tag{7}$$

because the upper semicontinuity of $t \mapsto C_{f_t}(x)$ implies that the functions (7) are upper and lower semicontinuous on T, respectively, and thus measurable on T.

Proof Let d be any element of $\mathbb{R}^n$. Then $\int_T f_t^{\uparrow}(x, \mathrm{d})\mu(dt)$ and $\int_T f_t^{\downarrow}(x, d)\mu(dt)$ are well-defined by Lemma 1. We assert that

$$\int_T f_t^{\downarrow}(x,d)\mu(\mathrm{d}t) \leq f^{\downarrow}(x,d) \leq f^{\uparrow}(x,d) \leq \int_T f_t^{\uparrow}(x,d)\mu(\mathrm{d}t). \tag{8}$$

To prove our assertion (8), we choose sequences $\alpha_i \downarrow 0$ and $\beta_i \downarrow 0$ such that

$$f^{\uparrow}(x,d) = \limsup_{\alpha_i \downarrow 0} \frac{f(x+\alpha_i d) - f(x)}{\alpha_i}$$

and

$$\liminf_{\beta_i \downarrow 0} \frac{f(x+\beta_i d) - f(x)}{\beta_i} = f^{\downarrow}(x,d),$$

respectively, which is possible since f is continuous on Ω by Lemma 2. Then, by the definition of f, we have

$$f^{\uparrow}(x,d) = \limsup_{\alpha_i \downarrow 0} \int_T \frac{f_t(x+\alpha_i d) - f_t(x)}{\alpha_i}\mu(\mathrm{d}t) \tag{9}$$

and

$$\liminf_{\beta_i \downarrow 0} \int_T \frac{f_t(x+\beta_i d) - f_t(x)}{\beta_i}\mu(\mathrm{d}t) = f^{\downarrow}(x,d). \tag{10}$$

Since f_t is Lipschitz of rank $k(t)$, it follows that

$$\left|\frac{f_t(x+\alpha_i d) - f_t(x)}{\alpha_i}\right|, \quad \left|\frac{f_t(x+\beta_i d) - f_t(x)}{\beta_i}\right| \leq \|d\| \cdot k(t).$$

Therefore, by using the hypothesis that k is in $L^1(T, \mathbb{R})$, and twice applying Theorem 1 in Preliminaries, we get the following inequalities:

$$\limsup_{\alpha_i \downarrow 0} \int_T \frac{f_t(x+\alpha_i d) - f_t(x)}{\alpha_i}\mu(\mathrm{d}t) \leq \int_T \limsup_{\alpha_i \downarrow 0} \frac{f_t(x+\alpha_i d) - f_t(x)}{\alpha_i}\mu(\mathrm{d}t) \tag{11}$$

and

$$\int_T \liminf_{\beta_i \downarrow 0} \frac{f_t(x+\beta_i d) - f_t(x)}{\beta_i}\mu(\mathrm{d}t) \leq \liminf_{\beta_i \downarrow 0} \int_T \frac{f_t(x+\beta_i d) - f_t(x)}{\beta_i}\mu(\mathrm{d}t). \tag{12}$$

Also, by the definition of $f_t^{\uparrow}$ and $f_t^{\downarrow}$, the following inequalities hold:

$$\int_T \limsup_{\alpha_i \downarrow 0} \frac{f_t(x+\alpha_i d) - f_t(x)}{\alpha_i}\mu(\mathrm{d}t) \leq \int_T f_t^{\uparrow}(x,d)\mu(\mathrm{d}t) \tag{13}$$

and

$$\int_T f_t^{\downarrow}(x,d)\mu(dt) \leq \int_T \liminf_{\beta_i \downarrow 0} \frac{f_t(x+\beta_i d) - f_t(x)}{\beta_i}\mu(\mathrm{d}t). \tag{14}$$

Hence, combining (14), (12), (10), (9), (11) and (13), we obtain

$$\int_T f_t^{\downarrow}(x,d)\mu(dt) \leq f^{\downarrow}(x,d) \leq f^{\uparrow}(x,d) \leq \int_T f_t^{\uparrow}(x,d)\mu(\mathrm{d}t)$$

and thus our assertion (8) follows.

Now, since $C_{f_t}(x)$ is compact for each t and the set-valued map $t \mapsto C_{f_t}(x)$ is upper semicontinuous on T, the functions

$$t \mapsto \max_{v_t \in C_{f_t}(x)} \langle v_t, d\rangle \quad \text{and} \quad t \mapsto \min_{w_t \in C_{f_t}(x)} \langle w_t, d\rangle$$

are upper and lower semicontinuous as functions of t, respectively, and hence measurable on T. Thus, by the definition of the convexificator $C_{f_t}(x)$, it follows that

$$\int_T f_t^{\uparrow}(x,d)\mu(\mathrm{d}t) \leq \int_T \max_{v_t \in C_{f_t}(x)} \langle v_t, d\rangle \, \mu(\mathrm{d}t) \tag{15}$$

and

$$\int_T \min_{w_t \in C_{f_t}(x)} \langle w_t, d\rangle \, \mu(\mathrm{d}t) \leq \int_T f_t^{\downarrow}(x,d)\mu(\mathrm{d}t). \tag{16}$$

Also, by the compactness of $C_{f_t}(x)$ and the definition of $C_f(x)$, the following inequalities hold:

$$\int_T \max_{v_t \in C_{f_t}(x)} \langle v_t, d\rangle \, \mu(\mathrm{d}t) \leq \max_{v \in C_f(x)} \langle v, d\rangle \tag{17}$$

and

$$\min_{w \in C_f(x)} \langle w, d\rangle \leq \int_T \min_{w_t \in C_{f_t}(x)} \langle w_t, d\rangle \, \mu(\mathrm{d}t). \tag{18}$$

Therefore, combining (18), (16), (8), (15) and (17), we conclude that

$$\min_{w \in C_f(x)} \langle w, d\rangle \leq f^{\downarrow}(x,d) \leq f^{\uparrow}(x,d) \leq \max_{v \in C_f(x)} \langle v, d\rangle .$$

Hence $C_f(x)$ is a convexificator of f at x and the set-valued map $x \mapsto C_f(x)$ is a (CF)-mapping of f as required. □

The following corollary gives convexificators of the integral functional, which are obtained by Clarke subdifferentials and Michel–Penot subdifferentials, respectively.

Corollary 1

(a) *Let $\partial_{cl} f_t(x) \subset \prod_n[-k(t), k(t)]$ be a convexificator of f_t at x, where k is in $L^1(T, \mathbb{R})$. Assume that the set-valued map $t \mapsto \partial_{cl} f_t(x)$ is upper semicontinuous on T. Then*

$$C_f(x) = \overline{co} \int_T \partial_{cl} f_t(x)\mu(dt)$$

is a convexificator of f at x and the set-valued-map $x \mapsto C_f(x)$ is a (CF)*-mapping of f.*

(b) *Let* $\partial_{mp} f_t(x) \subset \prod_n [-k(t), k(t)]$ *be a convexificator, and* $t \mapsto \partial_{mp} f_t(x)$ *be upper semicontinuous. Then the same result as (a) holds except using* $\partial_{mp} f_t(x)$ *instead of* $\partial_{cl} f_t(x)$.

Remark 4 The inclusion relation of the Clarke subdifferential of f to the integral of the Clarke subdifferential of each f_t is given in [2]. Also, the similar result for the Michel–Penot subdifferential holds in [1]. Our proposed integrated convexificator for f in Theorem 2 can be strictly contained in the Clarke subdifferential of f (see Examples 1 and 2 below). On the other hand, we note that the Michel–Penot subdifferential is in general only weak*-compact.

In the following examples, we specifically derive a convexificator of the integral functional.

Example 1 Let μ be the Borel measure on $T = [1, \infty)$. We define

$$f_t(x_1, x_2) = \frac{1}{t^2} \{|x_1| - |x_2|\}.$$

If we set

$$C_{f_t}(0) = co \left\{ \left(\frac{1}{t^2}, \frac{1}{t^2} \right), \left(-\frac{1}{t^2}, -\frac{1}{t^2} \right) \right\}$$

and $k(t) = \frac{\sqrt{2}}{t^2}$, then $C_{f_t}(0) \subset \prod_2 [-k(t), k(t)]$ is a convexificator of f_t at 0, k is in $L^1(T, \mathbb{R})$, and the set-valued map $t \mapsto C_{f_t}(0)$ is upper semicontinuous on T, and hence the hypotheses of Theorem 2 are satisfied. Therefore

$$\overline{co} \int_{[1,\infty)} C_{f_t}(0) \mu(dt) = co \{(1, 1), (-1, -1)\}$$

is a convexificator of f at 0, where $f(x_1, x_2) = |x_1| - |x_2|$. We note that the Clarke subdifferential of the integral functional f at 0 is

$$co \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}.$$

Example 2 Let μ be the counting measure on $T = (0, \infty)$. We define

$$f_t(x_1, x_2) = \begin{cases} \frac{1}{2^{t-1}} \{|x_1| - |x_2|\}, & t \in \mathbb{N}, \\ 0, & t \in T \setminus \mathbb{N}. \end{cases}$$

If we set

$$C_{f_t}(0) = \begin{cases} co \left\{ \left(\frac{1}{2^{t-1}}, \frac{1}{2^{t-1}} \right), \left(-\frac{1}{2^{t-1}}, -\frac{1}{2^{t-1}} \right) \right\}, & t \in \mathbb{N} \text{ and } t\text{: odd}, \\ co \left\{ \left(\frac{1}{2^{t-1}}, -\frac{1}{2^{t-1}} \right), \left(-\frac{1}{2^{t-1}}, \frac{1}{2^{t-1}} \right) \right\}, & t \in \mathbb{N} \text{ and } t\text{: even}, \\ \{(0, 0)\}, & t \in T \setminus \mathbb{N}. \end{cases}$$

and

$$k(t) = \begin{cases} \frac{\sqrt{2}}{2^{t-1}}, & t \in \mathbb{N}, \\ 0, & t \in T \setminus \mathbb{N}, \end{cases}$$

then $C_{f_t}(0) \subset \prod_2[-k(t), k(t)]$ is a convexificator of f_t at 0, k is in $L^1(T, \mathbb{R})$, and the set-valued map $t \mapsto C_{f_t}(0)$ is upper semicontinuous on T, and hence the hypotheses of Theorem 2 are satisfied. Therefore

$$\overline{co} \int\limits_{(0,\infty)} C_{f_t}(0)\mu(dt) = co\left\{\left(2, \frac{2}{3}\right), \left(\frac{2}{3}, 2\right), \left(-\frac{2}{3}, -2\right), \left(-2, -\frac{2}{3}\right)\right\}$$

is a convexificator of f at 0, where $f(x_1, x_2) = 2|x_1| - 2|x_2|$. We note that the Clarke subdifferential of the integral functional f at 0 is

$$co\,\{(2,2), (2,-2), (-2,2), (-2,-2)\}.$$

Remark 5 The convexificators of the integral functional f at 0 obtained in Examples 1 and 2 are strictly contained in the Clarke subdifferentials of f. This shows that certain results, such as mean value conditions and necessary optimality conditions that are expressed in terms of the proposed integrated convexificator, may provide sharp conditions.

Acknowledgement The author is thankful to the referee for his insightful and constructive comments.

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